In-plane elastic buckling of pin-ended shallow parabolic arches

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Abstract

The classical buckling theory is commonly adopted to evaluate the buckling load of arches regardless of their types and shapes. For shallow arches, however, the classical buckling theory may overestimate the buckling load because of a large pre-buckling deformation. In this study, the geometrically nonlinear behavior of pin-ended shallow parabolic arches subjected to a vertically distributed load is investigated to evaluate the buckling load. The nonlinear governing equilibrium equation of the parabolic arch is adopted to derive the buckling formula for a pin-ended shallow parabolic arch. Moreover, the threshold of different buckling modes (symmetric and asymmetric) is derived in terms of the slenderness ratio and the rise-to-span ratio of such arches. Numerical examples show that the proposed formula can accurately predict the buckling load of pin-ended parabolic arches.

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1. Introduction

Arches subjected to transverse loading may buckle in an in-plane buckling mode even if the out-of-plane buckling mode can be prevented by adequate lateral bracing. The in-plane buckling mode of arches can be symmetric or asymmetric, as shown in Fig. 1(a) and (b) respectively, where the symmetric mode is also called the symmetric snap-through mode. If the initial height of the arch is of the same order as the span, commonly referred to as deep arch, the buckling mode is asymmetric and the classical buckling theory can be used to estimate the in-plane buckling load. On the other hand, if the initial height is much smaller than the span, commonly referred to as shallow arch, the arch may buckle in a symmetric snap-through mode. In this case, the classical buckling theory may overestimate the buckling load because the ratio of the pre-buckling deformation to the height of the arch can be too large to be ignored. Therefore, to estimate the buckling load of shallow arches, the geometrically nonlinear behavior of the shallow arch needs to be considered.

The elastic in-plane buckling of arches was studied by Timoshenko and Gere [1], Vlasov [2], Austin [3], and Simitses [4]. They assumed that the arches were geometrically linear where the effect of pre-buckling deformation on the buckling was ignored. Several researchers conducted numerical analyses of the buckling of arches (Noor and Peters [5]; Calhoun and DaDeppr [6]; Elias and Chen [7]; Wen and Suhendro [8]). Recently, Pi and Trahair [9], and Moon et al. [10] reported that nonlinear pre-buckling deformation is significant to the buckling of shallow arches.

Early studies on shallow arches were performed by several researchers (Timoshenko and Gere [1]; Gjelsvik and Bondet [11]; Schreyer and Masur [12]; and Dickie and Broughton [13]). They derived analytical solutions of circular arches, but their solutions are limited to solid rectangular cross sections. Recently, Pi et al. [14], Bradford et al. [15], Rubin [16], and Chen and Lin [17–19] studied the buckling of shallow arches. In particular, Pi et al. [14] and Bradford et al. [15] provided solutions of both pin-ended and fixed circular shallow arches. They concluded that classical buckling theory overestimated both the symmetric snap-through and asymmetric buckling loads of shallow arches.
with dimensionless variables, the following parameters gives the dimensionless into Eq. 1. denotes the symmetric deformation, while the have been conducted. Austin [24], Gregory and Plaut [25], and Hsu [26], as shown in Fig. 2, can be written as

\[
EI \frac{d^4w}{dx^4} - \frac{EA}{2l} \int_0^l \left[ \left( \frac{dw}{dx} \right)^2 + 2 \frac{dw_0 dw}{dx} \right] dx \\
\times \left( \frac{d^2w_0}{dx^2} + \frac{d^2w}{dx^2} \right) - P(x) = 0
\]

where \(E\) = Young’s modulus; \(I\) = moment of inertia; \(A\) = area of the arch’s section; \(h\) = arch rise; \(l\) = arch span; \(w_0\) = vertical coordinate of the unloaded arch; \(w\) = deflection from the initial configuration; and \(P(x)\) = applied distributed load. To rewrite Eq. (1) with dimensionless variables, the following parameters are introduced:

\[
\xi = \frac{\pi}{l} x, \quad \eta_0 = \frac{w_0}{r}, \quad \eta = \frac{w}{r}, \quad q = \frac{P(x)}{EIr} \left( \frac{l}{\pi} \right)^4
\]

where \(r\) = radius of gyration of the cross section; \(\xi\) = dimensionless horizontal coordinate; \(\eta_0\) = dimensionless vertical coordinate of the unloaded arch; \(\eta\) = dimensionless deflection from the initial configuration; and \(q\) = dimensionless load. Substituting Eq. (2) into Eq. (1) gives the dimensionless nonlinear equilibrium equation:

\[
\frac{d^4\eta}{d\xi^4} - \frac{1}{2\pi} \int_0^\pi \left[ \left( \frac{d\eta}{d\xi} \right)^2 + 2 \frac{d\eta_0 d\eta}{d\xi} \right] d\xi \left( \frac{d^2\eta_0}{d\xi^2} + \frac{d^2\eta}{d\xi^2} \right) \\
- q = 0.
\]

For a parabolic arch, \(\eta_0\) and \(\eta\) are given as

\[
\eta_0 = H(\pi \xi - \xi^2), \quad \eta = \sum_{n=1}^N D_n \sin(n\xi)
\]

where \(H = 4h/\pi^2r\), the dimensionless rise of shallow parabolic arches. The assumed deformed shape \(\eta\) satisfies the boundary conditions of the pin-ended arch \([w(0) = w(l) = 0; w''(0) = w''(l) = 0]\). It is noted that the first term, \(D_1 \sin \xi\), of \(\eta\) in Eq. (4) denotes the symmetric deformation, while the second term, \(D_2 \sin 2\xi\), represents the asymmetric deformation.
Fig. 3. The assumed deformation shape of shallow parabolic arches: (a) the assumed symmetric deformation shape; (b) the assumed asymmetric deformation shape.

Fig. 3 shows that $D_1$ and $D_2$ are generalized coordinates of the symmetric and asymmetric deformation, respectively. In Eq. (4), modes higher than the second mode are not considered, because the buckling load of a higher mode is larger than those of the first two buckling modes.

Substituting Eq. (4) into Eq. (3) and using the Galerkin method gives

$$\frac{\pi^2}{2} \sum_{n=1}^{2} D_n n^4 \delta_{r n} + \left( \frac{1}{4} \sum_{n=1}^{2} D_n^2 n^2 + \frac{2H}{\pi} \sum_{n=1}^{2} D_n (1 - \cos r \pi) \right)$$

$$\times \left( \frac{\pi^2}{2} \sum_{n=1}^{2} D_n^2 n^2 \delta_{r n} - \frac{2H}{r} (\cos r \pi - 1) \right) + \frac{q}{r} (\cos r \pi - 1) = 0$$

(5)

where $\delta_{r n}$ denotes the Kronecker delta. Substituting $r = 1, 2$ into Eq. (5) leads to two separate equilibrium equations:

$$F_1(D_1, D_2, q) = \frac{\pi}{8} D_1^3 + 3H D_1^2 + \left( \frac{\pi}{2} + \frac{16H^2}{\pi} \right) D_1$$

$$+ 4H D_2^2 + \frac{\pi}{2} D_1 D_2^2 - 2q = 0$$

$$F_2(D_1, D_2, q) = 2\pi D_2^3 + 8\pi D_2 + \frac{\pi}{2} D_1^2 D_2$$

$$+ 8H D_1 D_2 = 0$$

(6)

where $F_1$ and $F_2$ are functions of the generalized coordinates $D_1, D_2$, and the dimensionless distributed load $q$.

3. Load–displacement relationship and buckling modes of pin-ended shallow parabolic arches

The incremental form is used to solve the nonlinear equation of the shallow parabolic arches. The dimensionless displacement and load increment, $\dot{d}_i$ and $\dot{\lambda}$ respectively, are defined as

$$d_i = D_i^{n+1} - D_i^n \quad (i = 1, 2; \quad n = 1, 2, 3, \ldots)$$

$$\lambda = q^{n+1} - q^n \quad (n = 1, 2, 3, \ldots)$$

(7)

where the superscript $n$ denotes the $n$th equilibrium path. The static perturbation method [23] is used to solve Eq. (6). The static perturbation method is applicable even if there is a singular point in the equilibrium path. When $D_1^n, D_2^n, q^n, D_1^{n+1}, D_2^{n+1}$ and $q^{n+1}$ are located on the equilibrium path, $F_r(D_1^n, D_2^n, q^n) = 0$ and $F_r(D_1^{n+1}, D_2^{n+1}, q^{n+1}) = 0$. Using the Taylor expansion, incremental form of separate equilibrium equation $Q_r$ can be expressed as

$$Q_r(d_1, d_2, \lambda) = F_r(D_1^{n+1}, D_2^{n+1}, q^{n+1}) - F_r(D_1^n, D_2^n, q^n)$$

$$= 2 \sum_{i=1}^{2} \frac{\partial F^n_r}{\partial D_i} \dot{d}_i + \frac{\partial F^n_r}{\partial \lambda} \dot{\lambda} + \frac{1}{2} \left[ \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2 F^n_r}{\partial D_i \partial D_j} \dot{d}_i \dot{d}_j + \frac{\partial^2 F^n_r}{\partial \lambda^2} \dot{\lambda}^2 \right] + H.O.T = 0.$$  

(8)

where higher-order terms in Eq. (8) are ignored, according to the static perturbation method [23].

Using the Maclaurin expansion, the dimensionless displacements $d_1$ and $d_2$ and the load increment $\lambda$ can be expressed as

$$d_i(t) = \ddot{d}_i t + \frac{1}{2} \dddot{d}_i t^2 + H.O.T \quad (i = 1, 2)$$

$$\lambda(t) = \dot{\lambda} t + \frac{1}{2} \ddot{\lambda} t^2 + H.O.T$$

(11)

where $t$ is an arbitrary variable. Dots in Eq. (11) denote differentiation with respect to $t$. In order to simplify the problem, the first- and second-order terms are considered. Substituting Eq. (11) into Eq. (10), $Q_r$ becomes a function of $t$. Coefficients of $t$ and $t^2$ should equal zero in order to satisfy $Q_r(t) = 0$ for an arbitrary value of $t$. Finally, two perturbation equations are obtained as

$$\sum_{i=1}^{2} k_{r i j} \ddot{d}_i = f_r \lambda$$

$$\sum_{i=1}^{2} k_{r i j} \dddot{d}_i + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} k_{r i j} \ddot{d}_i \dot{d}_j + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} k_{r i j} \dddot{d}_i \dot{d}_j \lambda - f_r \lambda \dot{\lambda}^2 = 0$$

(12a)

$$\sum_{i=1}^{2} k_{r i j} \dddot{d}_i + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} k_{r i j} \ddot{d}_i \dot{d}_j + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} k_{r i j} \dddot{d}_i \dot{d}_j \lambda - f_r \lambda \dot{\lambda}^2 = 0$$

(12b)
Eq. (12a) is the linearized equation of Eq. (10) that is adopted in this study to calculate the response of the pin-ended shallow parabolic arches. Based on Eqs. (6) and (9), Eq. (12a) can be expressed in a matrix form as

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix} = \begin{bmatrix} 2\lambda D_1 \\ 0 \end{bmatrix}$$  \hspace{1cm} (13)

where

$$k_{11} = \frac{3\pi}{8} D_1^{n^2} + 6H D_1^n + \left( \frac{\pi}{2} + \frac{16H^2}{\pi} \right) + \frac{\pi}{2} D_2^n$$

$$k_{12} = k_{21} = 8H D_2^n + \pi D_1^n D_2^n$$

$$k_{22} = 6\pi D_2^n + 8\pi + \frac{\pi}{2} D_1^n + 8H D_1^n.$$  \hspace{1cm} (14)

$k_{ij}$ in Eq. (14) is the tangential stiffness of pin-ended shallow parabolic arches under uniform compression at the $n$th equilibrium path. It should be noted that $k_{ij}$ is a function of $D_1^n$ and $D_2^n$, which are related to the symmetric and the asymmetric deformations, respectively. If all $k_{ij}$s are positive, regardless of $D_1^n$ and $D_2^n$, the structure is stable. However, arches become unstable when any of the $k_{ij}$s become negative. The buckling is symmetric when $k_{11}$ becomes zero first, while the buckling is asymmetric when $k_{22}$ becomes zero first. Based on Eq. (14), $k_{ij}$ is always positive when $H < \pi/4$; $k_{11}$ can be equal to or less than zero if $\pi/4 < H < 1.85$; and $k_{22}$ can also be equal to or less than zero if $H > 1.85$. The variation in $k_{ij}$ with respect to $H$ is investigated and the governing buckling mode with respect to the dimensionless rise $H$ is summarized as follows:

(a) $0 < H < \pi/4$: the arch is stable.
(b) $\pi/4 < H < 1.85$: the symmetric buckling mode governs.
(c) $H > 1.85$: the asymmetric buckling mode governs.

Variations in $k_{ij}$ of the shallow parabolic arches with $H$ are investigated. Figs. 4–6 show the variations in $k_{ij}$ with $H = 0.5$, 1.5, and 2.5, respectively. It is noted that $D_2^0 = -0.001H$ denotes the initial imperfection that induces the asymmetric deformation. In Fig. 4, all stiffness components are always positive when $H = 0.5$ and the buckling does not occur. In Fig. 5, where $H = 1.5$, $k_{11}$ become zero first and the symmetric buckling occurs, while in Fig. 6, where $H = 2.5$, $k_{22}$ reaches zero and the asymmetric buckling occurs.

The dimensionless rise $H$ is an important factor on which the buckling mode of the shallow parabolic arches depends. Similarly, Pi et al. [14] and Bradford et al. [15] suggested a modified slenderness ratio for classifying the buckling mode of circular arches. The modified slenderness ratio is defined as $\lambda_s = S^2/(4rR)$, where $R =$ radius of the circular arch and $S =$ arch length. The threshold for the classification of different buckling modes proposed in this study, those of Pi et al. [14] and Bradford et al. [15], are compared in Fig. 7. For this, $H$ in this study and the modified slenderness ratio $\lambda_s$ are converted into functions of slenderness ratio $l/r$. The $x$- and $y$-axis are the rise-to-span ratio $h/l$ and the slenderness ratio of the cross section $l/r$, respectively. It is found that the thresholds in this study, Pi et al. [14] and Bradford et al. [15] are almost identical to each other, regardless of the shape of the arches (circular or parabolic) and the loading conditions.

Fig. 8 shows the load–displacement relationships of pin-ended shallow parabolic arches in terms of $D_1$ and $q$, where three load–displacement relationships are plotted for different $H$s. For $H = 0.5$, the arch is stable. The arch with $H = 1.5$ buckles in the symmetric snap-through mode, while the asymmetric mode occurs when $H = 2.5$. The deformed shapes of the arches for $H = 1.5$ and $H = 2.5$ clearly show symmetric and asymmetric buckling modes, as shown in Fig. 9.
4. Buckling load of pin-ended shallow parabolic arches

In this section, an approximate symmetric buckling load for pin-ended shallow parabolic arches is derived from the nonlinear governing equilibrium equation of shallow arches. If an arch buckles in the symmetric snap-through mode, the buckling mode can be obtained using the first term, $D_1 \sin \xi$, of $\eta$ in Eq. (4). Substituting $\eta = D_1 \sin \xi$ into Eq. (2) gives

$$\frac{\pi}{8} D_1^3 + 3HD_1^2 + \left( \frac{16H^2}{\pi} + \frac{\pi}{2} \right) D_1 - 2q = 0.$$  \hspace{1cm} (15)

Eq. (15) represents the dimensionless symmetric load–displacement relationship of pin-ended shallow parabolic arches subjected to a vertically distributed load. Substituting $q$ in Eq. (4) and the vertical displacement at the crown $w_c$ into Eq. (15) gives

$$P = \frac{EI}{2} \left( \frac{\pi}{T} \right)^4 \left[ \frac{\pi}{8} \left( \frac{w_c}{r} \right)^3 + 3H \left( \frac{w_c}{r} \right)^2 \right] + \left( \frac{16H^2}{\pi} + \frac{\pi}{2} \right) \left( \frac{w_c}{r} \right).$$  \hspace{1cm} (16)

Substituting Eq. (17) into Eq. (16) gives

$$P_{cr, sym} = \frac{EI}{2} \left( \frac{\pi}{T} \right)^4 \times \left[ \frac{\pi}{8} D_1^{cr} + 3HD_1^{cr} + \left( \frac{16H^2}{\pi} + \frac{\pi}{2} \right) D_1^{cr} \right].$$  \hspace{1cm} (18)

where $P_{cr, sym}$ denotes the symmetric buckling load of the pin-ended shallow parabolic arches.

The compressive thrust in the arch can be obtained from Eq. (1):

$$N = \frac{EA}{2l} \int_{0}^{l} \left[ \left( \frac{dw}{dx} \right)^2 + 2 \frac{dw_0}{dx} \frac{dw}{dx} \right] dx.$$  \hspace{1cm} (19)

After substituting the assumed mode shape $\eta = D_1 \sin \xi$ and $D_1^{cr}$ into Eq. (19), the symmetric buckling compression,
referred to as $N_{cr, sym}$, is obtained as

$$N_{cr, sym} = \frac{\pi^2 EI}{l^2} \left( \frac{D_1^2 \pi^2}{4} + \frac{4H}{\pi} D_1^{cr} \right).$$

(20)

The strength of the shallow parabolic arches under uniform compression is governed either by the asymmetric buckling or the symmetric snap-through buckling. The asymmetric buckling compression is based on the classical buckling theory and can be expressed as

$$N_{cr, asym} = \frac{\pi^2 EI}{(\beta S)^2}$$

(21)

where $\beta$ = the buckling coefficient that depends on the boundary condition. The values of $\beta$ is 0.5 for pin-ended arches. As discussed before, the buckling mode depends on $H$. So, Eq. (20) or (21) should be taken accordingly as shown below:

$$N_{cr} = N_{cr, sym} \quad \text{for} \quad \pi/4 < H < 1.85$$

$$N_{cr} = N_{cr, asym} \quad \text{for} \quad H > 1.85$$

(22)

where $N_{cr}$ = the buckling compression. Fig. 10 shows the variation in $N_{cr}/N_{cr, asym}$ with respect to $H$. It is observed that $N_{cr}/N_{cr, asym}$ increases and approaches to one with increasing $H$ when $H < 1.85$. In other words, the classical buckling theory overestimates the buckling compression when the symmetric snap-through buckling occurs.

To verify the proposed buckling formula [Eq. (20)] and the threshold of different buckling modes, the result in this study is compared to those of finite-element analyses and Pi et al. [14], as shown in Fig. 10. The commercial program ABAQUS [27] was used for the finite-element analyses, where the arch is modeled by 50 three-node quadratic beam elements.

In Fig. 10 it is observed that the result using the proposed formula and the threshold show good agreement with those of finite-element analyses, except when $H$ is close to 1.85 (the maximum discrepancy is 13%). On the other hand, results of Pi et al. [14] generally underestimate $N_{cr}/N_{cr, asym}$. This is because the solution of Pi et al. [14] is derived only for the circular arches.

The discrepancy between the results from the proposed formula and the finite-element analysis can be explained as follows. When $H$ is close to 1.85, the interactive buckling [Fig. 11(c)] occurs where the buckling mode is a combination of the symmetric [Fig. 11(a)] and asymmetric [Fig. 11(b)] modes. Each buckling mode in Fig. 11 is marked in Fig. 12 for the corresponding regions. More than two terms in the assumed deflection function are needed to express the interactive buckling mode. However, only one term of the assumed deflection function, $\eta = D_1 \sin \xi$, is used to derive Eq. (20) in this study, which causes the disagreement.

5. Conclusion

The buckling of pin-ended shallow parabolic arches under vertically distributed loads is investigated to derive the threshold of different buckling modes (asymmetric and symmetric) and a buckling formula for the symmetric snap-through mode. The generic nonlinear equilibrium equation of an arch is adopted to derive the equation for the pin-ended shallow parabolic arch under uniform compression. The nonlinear equation is reformulated to an incremental form and the load–displacement relationship of the arch is obtained. The thresholds of different buckling modes are obtained from the stiffness terms in the load–displacement relationship and expressed in terms of the dimensionless rise $H$. The arch is
stable when $H < \pi/4$; symmetric snap-through buckling occurs when $\pi/4 < H < 1.85$; asymmetric buckling occurs when $H > 1.85$.

The symmetric buckling load is found by solving the nonlinear equilibrium equation using the assumed symmetric deformed shape. The proposed formula for the buckling load and the thresholds for the different buckling modes are verified by comparing with the finite-element method. The result using the proposed formula and the threshold show good agreement with those of finite-element analyses, except when interactive buckling occurs.

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