Lateral–torsional buckling of I-girders with discrete torsional bracings

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ABSTRACT

Discrete torsional bracing systems are widely used in practice to increase the lateral–torsional buckling (LTB) strength of I-girders. However, only limited studies are available on the LTB strength of I-girders with mid-span torsional bracing. In addition, equivalent continuous brace stiffness concept is adopted for general discrete torsional bracing problems. This article presents an analytical solution for LTB strength and stiffness requirements of I-girders with discrete torsional bracings under a uniform bending condition. Firstly, the critical moment and torsional stiffness requirement are derived by using an energy method for an arbitrary number of bracing points. The proposed equations are then compared with the results of finite element analyses and those obtained by previous researchers. From the results, it is found that the proposed solutions agree well with the results of finite element analyses regardless of the number of bracing points, while the results for the equivalent continuous brace stiffness concept are not suitable for multiple discrete torsionally braced beams. Finally, reduced formula for the total stiffness requirement is proposed for the purpose of design, and effects of linear moment gradient loading and geometric imperfections on critical moments and stiffness requirement are also observed.

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1. Introduction

The lateral–torsional buckling (LTB) strength of an I-girder can be increased by using an adequate bracing system. Bracing systems can be divided into two categories: lateral bracing and torsional bracing. In particular, torsional bracing prevents the twisting of the section so that the LTB strength can be improved. Some examples of torsional bracing are shown in Fig. 1. Torsional bracing can be divided further into two categories: continuous and discrete torsional bracing. A deck attached to the top flange and a cross frame are typical examples of continuous and discrete torsional bracing, respectively, as shown in Fig. 1.

For the LTB strength of beams with continuous torsional bracing, Taylor and Ojulso [1] developed the exact solution for the critical moment of beams. Their solution is adapted for discrete torsional bracing by summing the stiffness of each bracing along the span and dividing by the beam length to obtain an equivalent continuous bracing stiffness [2,3]. Several studies have been carried out on the LTB of beams with discrete torsional bracing. Trahair [3] proposed a stiffness requirement for beams with a mid-span torsional restraint based on a numerical approximation. Valentino and Trahair [4] performed several solutions for beams with mid-span torsional bracing under various loading conditions. Valentino et al. [5] studied the effects of mid-span torsional restraints on inelastic buckling by using the approximated solution proposed by Trahair [3]. Also, Mutton and Trahair [6], Nethercot [7], Medland [8], and Tong and Chen [9] have studied the effects of torsional bracing on the LTB strength of beams for various loading conditions. However, almost all their studies are limited to mid-span torsional bracing problems. Thus, it seems that a significant amount of study is still required for torsionally braced beams, especially for those with multiple bracing points.

This article presents an analytical solution for the LTB strength and stiffness requirement of I-girders with discrete torsional bracings. I-girders are considered to be simply supported in flexure and torsion and have a doubly symmetric section. Firstly, the critical moment and torsional stiffness requirement are derived by using the Rayleigh–Ritz method [10,11] for an arbitrary number n of bracing points so that the critical moment and torsional stiffness requirement are a function of n. The proposed equations are then compared with the results of finite element analyses B.A.S.P. [12] and those obtained by previous researchers [2,3]. From the results, it is found that the proposed equations show good correlation with finite element analyses regardless of the number of bracing points and are successfully verified, while results show that the equivalent continuous brace stiffness concept is not suitable for the multiple discrete torsionally braced beam. Finally, a reduced formula for the total stiffness requirement is proposed for the purpose of design.
2. Lateral–torsional buckling analysis of I-girder with discrete torsional bracings

2.1. Critical moment

Fig. 2 shows the I-girder with discrete torsional braces. The boundary condition of the girder is simply supported in flexure and torsion and the cross-section is doubly symmetric. The I-girder is subjected to two equal end moments \( M_s \) and is equally spaced by \( n \) number of torsional bracings. Thus, the length between bracing points is \( L/(n + 1) \), where \( L \) is the whole length of the I-girder. The coordinate system is also shown in Fig. 2. The \( x \), \( y \), and \( z \) axes denote the out-of-plane, in-plane, and longitudinal direction of the girder, respectively. The corresponding deformations to the \( x \), \( y \), and \( z \) axes are \( u \), \( v \), and \( w \), respectively. The rotation of the cross-section is represented by the twisting angle \( \phi \) and the torsional bracings are considered as elastic restraints represented by torsional stiffness \( R \) as shown in Fig. 2.

In this study, the LTB strength and stiffness requirement of the I-girder shown in Fig. 2 are derived using the total potential energy principle \([10,11]\). For the derivation of the solutions, the following assumptions are used: (a) the deformation of the member is small enough; (b) the distortion of the cross-section is neglected; (c) the material follows Hook’s law; (d) the elastic restraints are attached to the centroidal axis; and (e) the effect of in-plane deformation can be ignored. Based on these assumptions, the total potential energy of the I-girder can be expressed as

\[
\Pi = \int_0^l \frac{1}{2} \left( EI_u u''^2 + EI_v v''^2 + GJ \phi''^2 - 2M_s u' \phi \right) dz + \frac{1}{4} R \sum_{i=1}^n \phi_i^2
\]

(1)

where \( E \) is Young’s modulus; \( G \) is the shear modulus of elasticity; \( I_y \) is the second moment of inertia about the y axis; \( I_w \) is the warping constant; \( f \) is the pure torsional constant; and \( \phi_i \) is the twisting angle of the cross-section at bracing points. The I-girder is equally spaced at \( n \) number of torsional bracings. Thus, the location of the \( i \)th torsional bracing can be defined as \( z = (i/n + 1)L \), where \( (i = 1, 2, 3, \ldots, n) \).

For a simply supported girder in flexure and torsion, the out-of-plane deformation \( u \) and the twisting angle \( \phi \) can be assumed as

\[
u = \sum_{k=1}^{n+2} u_k \sin \frac{k\pi z}{L} \quad \text{and} \quad \phi = \sum_{k=1}^{n+2} \phi_k \sin \frac{k\pi z}{L}
\]

(2)

where \( u_k \) and \( \phi_k \) are generalized coordinates for the lateral deformation and twisting angle, respectively. \( n + 1 \) buckling modes are considered to provide the critical moment corresponding to the lowest buckling mode when the girder is fully restrained. The assumed deformed shapes satisfy the following boundary conditions of the girder: (a) \( u(0) = \phi(0) = u(L) = \phi(L) = 0 \); (b) \( u''(0) = \phi''(0) = u''(L) = \phi''(L) = 0 \). Substituting Eq. (2) into Eq. (1), the total potential energy can be simplified in terms of \( u_k \) and \( \phi_k \) as

\[
\Pi = \frac{\pi^4 EI}{4L^2} \sum_{k=1}^{n+2} k^4 u_k^2 + \frac{\pi^4 EI}{4L^3} \sum_{k=1}^{n+2} k^2 \phi_k^2 \quad + \frac{\pi^2 GJ}{4L} \sum_{k=1}^{n+2} k^2 \phi_k^2 + \frac{\pi^2 M}{2L} \sum_{k=1}^{n+2} k^2 u_k \phi_k + \frac{1}{2} R \sum_{i=1}^n \phi_i^2.
\]

(3)

For equilibrium, the first variation of the total potential energy must vanish so that \( \delta \Pi/\delta u_k = 0 \) and \( \delta \Pi/\delta \phi_k = 0 \), where \( k = 1, 2, 3, \ldots, n + 2 \). Also, these relationships can be expressed in a matrix form as

\[
\begin{bmatrix}
[K_{11}] & [K_{12}] \\
[K_{21}] & [K_{22}]
\end{bmatrix}
\begin{bmatrix}
\{u_k\} \\
\{\phi_k\}
\end{bmatrix}
= \begin{bmatrix}
\{0\} \\
\{0\}
\end{bmatrix}.
\]

(4)

Therefore, the critical moment can be obtained from nontrivial solutions of Eq. (4) and are given as Eqs. (5a)–(5c) and (6) in Box 1. In Eq. (5), \( M_{cr,m}, M_{cr,n} \) and \( M_{cr,(n+1)} \) represent the critical moment corresponding to the \( m \), \( n \), and \( (n + 1) \)th buckling modes, respectively. In addition, \( W \) is the torsional slenderness defined as \( (\pi/L)\sqrt{EI_u/GJ} \). It is noted that Eq. (5a) is only available when the number of bracing points \( n \) is larger than 1. For example, for the I-girder with a mid-span torsional bracing problem \( (n = 1) \), critical buckling moments for symmetric and asymmetric buckling modes can be obtained from Eqs. (5b) and (5c), respectively, whereas Eq. (5a) is not used for calculation.

It is also noted that Eq. (5a) is available when \( R \leq \sum R_m \) and Eq. (5b) can be adopted to calculate the critical moments when \( \sum R_m < R \leq R_T \), while Eq. (5c) is used to calculate the critical moment when \( R > R_T \), where \( R_m \) and \( R_T \) are the stiffness required to change the buckling configuration from the \( m \)th mode to the \( (m + 1) \)th mode and total stiffness which provides full bracing.
respectively. Fig. 3 shows the variation in critical moment with torsional stiffness R. It can be seen that the critical moment increases with increasing R and the (m + 1)th mode occurred when $R > \sum R_m$. When $R > R_T$, full bracing is provided, and the critical moment is then the same as $M_{cr,m+1}$. The method used to calculate $\sum R_m$ and $R_T$ is discussed in the next section.

### 2.2. Stiffness requirement

The required stiffness, which changes the buckling configuration from the mth mode to the (m + 1)th mode $R_m$, can be obtained in increments of critical moment from $M_{cr,m}$ to $M_{cr,m+1}$.

Then, $\sum R_m$ is simplified as

$$\sum R_m = \left( \frac{L}{(n+1)E_I} \right) \sum_{m=1}^{n-1} (M^2_{cr,m+1} - M^2_{cr,m})$$  \hspace{1cm} (7)

where $\sum R_m$ is the summation of required stiffness $R_m$. Expanding the right term in Eq. (7) gives

$$\sum R_m = \frac{\pi^2 GJ}{L} (n-1) \left[ 1 + \left( n^2 + 1 \right) W^2 \right].$$  \hspace{1cm} (8)

Similarly, from Eqs. (5b) and (5c), the required stiffness which changes the buckling configuration from the mth mode to (n + 1)th...
mode, \( R_n \), can be computed in the last increment of the critical moment form \( M_{cr,n} \) to \( M_{cr,(n+1)} \). Then, \( R_n \) is obtained by

\[
R_n = \frac{\pi^2 G J}{L} \left( a_n W^n + b_n W^n + c_n \right)
\]

where

\[
a_n = 16(n + 1)^6 - 4(n + 1)^4 + 4(n + 1)^2 - 1;
\]

\[
b_n = 16(n + 1)^4 + 4(n + 1)^2 - 2
\]

\[
c_n = 4(n + 1)^2 - 1; \quad d_n = 6(n + 1)^2 + 1.
\]

Finally, the total stiffness requirement \( R_T \) can be determined by the summation of \( \sum R_m \) in Eq. (7) and \( R_n \) in Eq. (9) as given below

\[
R_T = \frac{\pi^2 G J}{L} \left( \sum f_1(n) W^n + f_2(n) W^n + f_3(n) \right)
\]

where

\[
f_1(n) = n - 1
\]

\[
f_2(n) = 28(n + 1)^6 - 48(n + 1)^5 + 70(n + 1)^4 - 56(n + 1)^3 + 16(n + 1)^2 - 8(n + 1) - 1
\]

\[
f_3(n) = 30(n + 1)^4 - 32(n + 1)^3 + 18(n + 1)^2 - 12(n + 1) - 2
\]

\[
f_4(n) = 6(n + 1)^2 - 4(n + 1) - 1
\]

\[
f_5(n) = 6(n + 1)^2 + 1
\]

Thus, the LTB moment of the girder with discrete torsional bracings for an arbitrary number of bracing points \( n \) can be calculated form Eq. (5) with Eqs. (9)–(12) according to the following procedure: (a) computing of \( \sum R_m \) and \( R_T \) with Eqs. (9)–(12); (b) comparing \( R \) with \( \sum R_m \) and \( R_T \); and (c) calculating the corresponding critical moment from Eq. (5).

### 3. Verification, comparative study, and design recommendation

#### 3.1. FEM model description

In this study, the proposed solutions are compared with those of finite element analyses and obtained by other researchers for the purpose of verification. Eigen-value analyses are performed by using the B.A.S.P program [12]. Various I-girder models are selected based on web and flange proportions to prevent local buckling according to the AISC specification [13]. The analyzed beams are fully stiffened by transverse stiffeners attached at the bracing points with equal heights and widths of the beam sections to avoid cross-section distortions. The thickness of the stiffeners is 15 mm, and the unbraced lengths of the girders vary from 3000 mm to 5000 mm. Detailed profiles of analysis models are given in Table 1.

A uniform bending moment is applied at each end and is expressed as the tension and compression forces at the bottom flange and the top flange, respectively, as shown in Fig. 4. Point A is a hinged end where the displacement in directions \( x, y, z \) and the rotation about the \( z \) axis are restrained, while point B is a roller end where the displacement in directions \( x, y \) and the rotation about the \( z \) axis are restrained. The displacement in direction \( x \) for lines \( a \) and \( b \) is restrained. Lines \( c \) and \( d \) are restrained in direction \( y \). The torsional braces can be modeled as rotational spring elements with a rotational stiffness \( R \). The rotational spring elements are evenly attached at the bracing points along the centroidal line of beams.

The required stiffness can be obtained by increasing the spring stiffness until the beam buckles between bracing points. When full bracing is provided, the critical moment remains constant with increasing restraint stiffness. The evaluations of stiffness requirements for full bracing (total stiffness requirement) are performed for all analysis models shown in Table 1, while variations in critical buckling moments with \( R \) are illustrated using analysis results of the M1 model.

#### 3.2. Results obtained by previous researchers

Trinhair [3] suggested a stiffness requirement for beams with a mid-span torsional restraint based on the numerical approximation as follows

\[
R_T \approx \frac{GJ}{L} (30 + 100W^2)
\]

where \( R_T \) is the approximate required stiffness. The contribution of torsional restraint stiffness was also presented using the approximation method [7]. This solution was then adapted to evaluate critical buckling moments of beams with a mid-span torsional brace.

### Table 1

Profiles of analysis models.

<table>
<thead>
<tr>
<th>Model</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
<th>M10</th>
<th>M11</th>
<th>M12</th>
<th>M13</th>
<th>M14</th>
<th>M15</th>
<th>M16</th>
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<td>340</td>
<td>300</td>
<td>300</td>
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<td>220</td>
<td>220</td>
<td>200</td>
<td>200</td>
<td>180</td>
<td>180</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td>( t_y ) (mm)</td>
<td>24</td>
<td>24</td>
<td>20</td>
<td>20</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>16</td>
<td>16</td>
<td>16</td>
<td>16</td>
<td>14</td>
<td>14</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>( h_t ) (mm)</td>
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<td>976</td>
<td>930</td>
<td>880</td>
<td>832</td>
<td>782</td>
<td>732</td>
<td>682</td>
<td>634</td>
<td>584</td>
<td>534</td>
<td>484</td>
<td>436</td>
<td>386</td>
<td>336</td>
<td>286</td>
</tr>
<tr>
<td>( t_w ) (mm)</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
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<td>10</td>
</tr>
<tr>
<td>( L_* ) (mm)</td>
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<td>5000</td>
<td>4500</td>
<td>4500</td>
<td>4250</td>
<td>4250</td>
<td>4000</td>
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<td>3500</td>
<td>3250</td>
<td>3250</td>
<td>3000</td>
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</tr>
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</table>

Fig. 3. Illustration of critical moments with increasing torsional stiffness.

Fig. 4. Loading and boundary conditions of analysis models.
Fig. 5. Stiffness requirement for beams with a mid-span torsional bracing ($n=1$).

while the required stiffness is provided [3,9]. The equation is given as

$$\left( \frac{M_{cr}}{M_{ocr}} \right)^2 = 1 + \frac{R}{R_T} \left( \frac{M_{cr,ld}}{M_{ocr}} - 1 \right)$$

(14)

where $M_{cr}$ is the critical moment corresponding to an arbitrary torsional stiffness $R$; and $M_{cr,ld}$ is the critical moment when full bracing is provided.

The contribution of torsional bracings to the buckling strength of a beam with a continuous torsional bracing was derived by Taylor and Ojalvo [1]. This was modified by Yura [2], who proposed that the required stiffness was determined by considering the discrete torsional braces as an equivalent continuous torsional bracing. Trahair [3] also provided the same ideas by treating discrete restraints as an equivalent uniform continuous restraint. The relationship between critical moment and restraint stiffness is simplified as

$$M_{cr} = \sqrt{M_{ocr}^2 + R_T E I}$$

(15)

where $R_T$ is an equivalent continuous torsional bracing stiffness which is defined as $R_T = n R_T'$.

By substituting $R_T$ into Eq. (15) then rearranging Eq. (15), the required stiffness $R_T'$ can be obtained as

$$R_T' = \frac{1}{n} \left( \frac{M_{cr}^2 - M_{ocr}^2}{E I} \right)$$

(16)

Comparisons between the above results obtained by previous researchers and the proposed solutions and finite element results are presented in the following section.

3.3. Verification and comparison

Fig. 5 shows the comparisons of total stiffness requirements for beams with a mid-span torsional restraint ($n=1$). The results from Eq. (11) agree well with those from finite element analyses, while Eq. (16) generally overestimates the required stiffness. Eq. (13) also shows good correlation with a 2% error. These results, therefore, reveal that the equivalent continuous torsional bracing concept is not suitable for a discrete brace system and it is known that the results obtained by Trahair [3] and the proposed solution provide an adequate estimation of total stiffness requirements for beams with a mid-span torsional bracing. The variations in $M_{cr}/M_{ocr}$ or $M_{cr,FEM}/M_{ocr}$ with $R/R_T$ are shown in Fig. 6. It can be seen that the solution of this study, Trahair [3], and the finite element analyses are almost identical with each other and the results of this study are successfully verified, while Eq. (15) provides a poor correlation with the finite element analysis results due to the overestimation of $R_T$, as shown in Fig. 5.

The verifications and comparisons for beams with multiple torsional bracings are also performed in this study. The comparison results of the total stiffness requirement of beams with 2 and 3 torsional braces ($n=2$ and 3) with the same braced lengths are shown in Figs. 7 and 8, respectively. It was found that the proposed solutions and the equivalent continuous torsional bracing concept provide the correct total stiffness requirement $R_T$ for $n=2$, while the equivalent continuous torsional bracing concept understimates the total stiffness requirement $R_T$ for $n=3$. The variation in critical moment with $R/R_T$ is shown in Figs. 9 and 10 for $n=2$ and 3, respectively. Even though the equivalent continuous torsional bracing concept shows a poor relation for $R_T$ when $n=3$, the critical moment is similar to those obtained in this study as shown in Fig. 10. It is also seen in Figs. 6, 9 and 10 that the equivalent continuous torsional bracing concept shows better relations with the finite element analyses results with increasing $n$. 

Fig. 6. Buckling moments of beams with a mid-span torsional bracing ($n=1$).

Fig. 7. Stiffness requirement for beams with 2 torsional bracings ($n=2$).

Fig. 8. Stiffness requirement for beams with 3 torsional bracings ($n=3$).
The comparison of $R_T$ is extended to $n = 5$ and the results are shown in Fig. 11. It is found that the proposed equations show good correlation with the finite element analyses regardless of the number of bracing points and were successfully verified, while the results of the equivalent continuous brace stiffness concept significantly underestimate the torsional stiffness requirement when the number of bracing points is larger than 3. Thus, again, the equivalent continuous torsional bracing concept is not suitable for estimating the $R_T$ of discrete torsionally braced beams, while the proposed solutions can be applied with good accuracy. The comparison of critical moments is also extended to $n = 5$ and the results are shown in Fig. 12. The $x$ and $y$ axes in Fig. 12 represent the number of bracing points $n$ and the maximum error between the results of the finite element analyses and those of this study or the equivalent continuous torsional bracing concept. It is found that the results of this study are well matched with those of the finite element analyses with good accuracies, while the results from the equivalent continuous torsional bracing concept show poor correlation when $n$ is small. This means that Eq. (15) is not suitable when the number of discrete torsional bracings is small.

### 3.4. Reduced formula for total stiffness requirement

The critical moment when full bracing is provided can be defined as

$$M_{cr,lb} = \left( \frac{\pi}{L_b} \right)^2 EI_w J_l \left( 1 + \left( \frac{\pi}{L_b} \right)^2 \frac{EI_w}{GJ_w} \right)$$  \hfill (17)

where $L_b$ is the length between the bracing points. To obtain a reduced formula of the total stiffness requirement $R_T$, Eq. (11) can be rearranged in terms of $M_{cr,lb}$ as

$$R_T = \frac{M_{cr,lb}^2 L_b}{EI_w} K_w.$$  \hfill (18)

where

$$K_w = \frac{1}{2 f_3^2(n)} \left( f_2(n) w^4 + f_3(n) W^2 + f_4(n) \right)$$  \hfill (19)

and $f_1(n) - f_2(n)$ are denoted in Eq. (12).

$K_w$ is a function of torsional slenderness $W$ and the number of bracing points $n$. If $W$ approaches infinity, $K_w$ may reach an asymptotic value. An asymptotic value of $K_w$ is defined as $K_0$ and given by

$$K_n = \frac{f_2(n)}{2 f_3^2(n)} f_5(n).$$  \hfill (20)

$K_n$ is the only function of $n$. The variation of $K_w/K_n$ with $W$ is shown in Fig. 13. It is found that $K_w/K_n$ approaches 1.0 with an increasing $W$ regardless of $n$. This means that $K_w$ can be replaced by $K_n$ if $W$ is large enough. From Fig. 13, it can be seen that $K_w/K_n$ approaches 1.0 when $W \geq 0.5$ for $n = 1$ and $W \geq 0.1$ for $n = 10$.

For the I-girder, the torsion is mainly resisted by warping so that the warping constant is normally much larger than the pure torsion constant. Based on the existing highway bridge profiles, the ranges for the overall dimensions of the investigated main girders are as follows: (a) overall depth: $L/18 \leq D \leq L/8$; (b) web thickness: $t_w \geq D/150$; (c) flange width: $D/6 \leq b_f \leq D/3$ or $b_f \geq L/85$; (d) flange thickness: $t_f \geq 1.1 t_w$ or $b_f/24 \leq t_f \leq b_f/10$; and (e) slenderness ratio: $L/r_y \leq 300$. Various I-section profiles are defined based on the cross-section proportions in order to observe the distribution of $W$. The span lengths are first chosen to vary from 4 m to 50 m. Then, the height of the web and the width of the flanges are analogously defined and adjusted to satisfy the limiting slenderness ratio. In this observation, the slenderness ratios of all sections approximately reach 300. As shown in Fig. 14, about 79%...
of the cross-section profiles are located in the range of $1.00 \leq W \leq 1.50$, and the remaining 21% are distributed within 0.50 and 1.00. It is clear that almost all the values of $W$ are larger than 0.50. Thus, Eq. (20) can be used to calculate the $R_T$ instead of $K_w$.

4. Effects of loading conditions and initial imperfections

4.1. I-girder with discrete torsional bracings under linear moment gradient

The critical moment of fully restrained beams under linear moment gradient can be expressed as

$$M_{cr,\alpha} = C_b M_{cr,\alpha} M_b$$

where $M_{cr,\alpha}$ is the critical moment of I-girder under linear moment gradient; $M_{cr,\alpha}$, $M_b$ is the critical moment of I-girder under uniform bending; and $C_b$ is the moment gradient factor. Based on the numerical investigation, in this study, $C_b$ for I-girder with discrete torsional bracings is suggested and can be approximated as

$$C_b = 1 + (1 + \alpha) f_n \leq 1.55$$

where $\alpha$ is an end moment ratio and has a positive value for double curvature bending, while it has a negative value for single curvature bending. $f_n$ is a function of an arbitrary number of bracing points $n$ and given as

$$f_n = 0.0125n^2 - 0.1314n + 0.46.$$  

A comparison of the proposed values of $C_b$ calculated from Eq. (22) with those obtained from finite element analyses for various values of $\alpha$ is shown in Fig. 15. It is found that the moment gradient factor $C_b$ is proportional to an end moment ratio when $n$ is larger than 1. However, for I-girder with a mid-span torsional restraint, the relationship is nonlinear and the limiting value of 1.55 is proposed for $C_b$ in order to provide a conservative determination of critical moments. From Fig. 15, it is also observed that the results from Eq. (22) are almost identical with those from finite element analyses regardless the number of bracing points. Thus, Eq. (22) can be used when the I-girder are fully restrained.

Fig. 16 shows the variation in critical moment obtained from finite element analyses with torsional stiffness for I-girder with three torsional restraints ($n = 3$). The critical moments of I-girder under uniform bending $M_{cr,\alpha} = 1.00$ are chosen as benchmark values in order to compare with critical moment of other loading cases. It is found that the critical moments almost remain constant when required stiffness is provided. Therefore, the stiffness requirement for the I-girder under linear moment gradient can be determined using Eq. (11) which is derived for I-girder under uniform bending for the conservative design purpose.

4.2. Effects of initial imperfections

The critical moment may be significantly reduced due to the effects of initial imperfections. For an example of effects of initial imperfection, geometric nonlinear analyses are conducted for model $M1$ with a mid-span torsional restraint ($n = 1$) and three restraining points ($n = 3$). A computer program ABAQUS [14] is adopted to investigate for two cases of initial imperfections with magnitudes of $L_b/750$, and $L_b/1500$, respectively. The variation of critical moments and restraint stiffness are illustrated in Figs. 17 and 18. It is clear that the critical moments are decreasing due to the effects of initial imperfections. In the case of initial imperfection with $L_b/750$, the critical moments decrease about 10% of buckling strength regardless the number of torsional restraints, while for $n = 1$ it is found that the reduction of critical moment is larger than the case of $n = 3$. 

![Fig. 13. Variation in $K_w/K_n$ with $W$.](image1)

![Fig. 14. Distribution of $W$ in cross-section design.](image2)

![Fig. 15. Moment factor $C_b$ with various end moment ratios.](image3)

![Fig. 16. Comparison of critical moments with various end moment ratios.](image4)
Fig. 17. Variation of critical moments with increasing torsional stiffness \((n = 1)\).

Fig. 18. Variation of critical moments with increasing torsional stiffness \((n = 3)\).

From Figs. 17 and 18, it can be seen that the stiffness requirement from finite element analyses is slightly larger than the theoretical value, which is for perfectly straight I-girder. However, it is hard to determine the exact values of stiffness requirement for I-girder with initial imperfection based on the results of this study. Thus, more intensive study for the effects of initial imperfection to the critical moment and the stiffness requirement are necessary for rigorous design of I-girder with discrete torsional bracing.

5. Conclusions

This article presents an analytical solution for the LTB strength and stiffness requirement of I-girders with discrete torsional bracings. Firstly, critical moments are derived based on the energy method as shown in Eq. (5). The required stiffness, which changes the buckling configuration to \((m + 1)\)th mode \(\sum R_m\) and the total stiffness requirement \(R_T\) are then proposed as shown in Eqs. (8) and (11), respectively. The proposed solutions are applicable for an arbitrary number of bracing points \(n\).

The proposed solutions are compared with the results of the finite element analyses and those obtained by previous researchers. From the comparison results, the proposed equations are successfully verified and it is found that the equivalent continuous brace stiffness concept is not suitable for calculating the total stiffness requirement \(R_T\). In particular, the equivalent continuous brace stiffness concept significantly underestimated the total torsional stiffness requirement when the number of bracing points is larger than 3. Also, the critical moment obtained from the equivalent continuous brace stiffness concept is not well matched with the results of the finite element analyses and those of this study when \(n\) is small, while the solution proposed in this study agrees well with the results of the finite element analyses, regardless of the number of bracing points.

The proposed total stiffness \(R_T\) is modified to a simple form and a reduced formula for \(R_T\) is suggested as shown in Eqs. (18) and (20) for the design purpose. The reduced formula is well matched with the results from Eq. (11) when the torsional slenderness is larger than 0.5. Finally, the effects of the linear moment gradient and initial imperfections are investigated. For fully restrained I-girder, the moment gradient factor \(C_b\) is suggested as Eq. (22) based on the results of the numerical investigation. It can also be seen that more numerical or experimental studies are needed to consider the effects of initial imperfection to the critical moment and the stiffness requirement for rigorous design of I-girder with discrete torsional bracing.

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